Gravitational interactions in the Earth-Jupiter-Sun System

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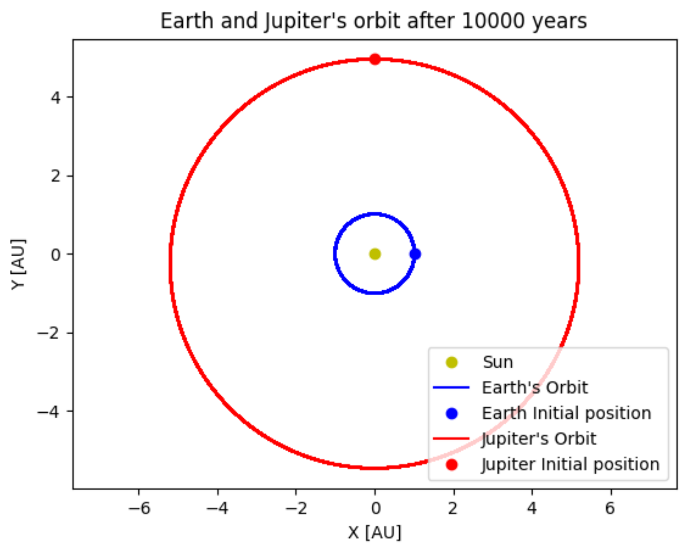

Abstract

In this project we investigate the Earth-Jupiter-Sun System and observe the effect Jupiter has on Earth's orbit. We do this by plotting two separate 2-body problems (Earth-Sun, Jupiter-Sun) so we can analyse the effects on Earth's orbit when we plot the 3-body problem. Mainly the changes in the perihelion/aphelion positions and also calculating the semimajor axis of the orbit and what effect that has on the eccentricity of the orbit.

1. <u>Background</u>

In 1911 a young Serbian mathematician, by the name of Milutin Milankovitch, began plotting the path of the ice ages during the Pleistocene epoch, seeking information on whether the change in Earth's Orbit, Axial positioning or spin had any correlation. Over the next thirty years, Milutin meticulously calculated the small orbital changes caused by the gravitational pull from other planets, creating patterns and relaying these patterns back roughly 600000 prior to the year 1800.

Releasing his results in 1941 in a book titled 'Canon of Insolation of the Earth and Its Application to the Problem of the Ice Ages', Milankovitch predicted, among other things, that the earths elliptical orbit around the Sun changed over time, estimating a period of roughly 96000 years. [1]

This change in Earths orbit is now known as the Milankovitch cycle and is thought to be mainly caused by the gravitational pull of the two largest planets in our Solar System: Saturn and Jupiter. Each planet is separately affected by these gravitational pulls and thus by separating them into three body problems, (in this case the Sun, Earth and Jupiter), we can observe the isolated impact each planet has on another planets orbit.

Unfortunately, it is impossible to find an analytic solution for 3 gravitational objects in motion, and hence many physicists of the late 1600's and early 1700's (mostly Isaac Newton) developed numerical methods to help describe the motion of the universe.

2. <u>Intro</u>

The two-body orbital system was originally solved by Isaac Newton in 1687 [2]. It can be done by changing it into two one-body problems, which can be solved exactly.

If we let R be the position of the Centre of Mass (CM), then we have that:

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = (m_1 + m_2) \ddot{R}$$

By newtons third law [3] we get that:

$$m_1 \ddot{x}_1 = -m_2 \ddot{x}_2$$

Hence this shows us that:

$$\frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{(m_1 + m_2)} = \ddot{R} = 0$$

Which shows us that the velocity $V = \frac{dR}{dt}$ of the centre of mass is constant, meaning the momentum is also constant and that the position of the centre of mass will always be able to be calculated from the initial positions and velocities.

Now let r be the displacement vector between masses 1 and 2, we have that:

$$\ddot{r}=\ddot{x}_1-\ddot{x}_2=\left(\frac{m_1\ddot{x}_1}{m_1}-\frac{m_2\ddot{x}_2}{m_2}\right)=\left(\frac{1}{m_1}+\frac{1}{m_2}\right)m_1\ddot{x}_1$$

Hence, we get that:

$$F(r)=\mu\ddot{r}\,,\, where\ \mu=\frac{1}{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}$$

Now once R(t) and r(t) are determined, we can find the solution to the two-body problem such that:

$$x_1(t)=R(t)+\frac{m_2}{m_1+m_2}r(t)$$

$$x_2(t)=R(t)-\frac{m_1}{m_1+m_2}r(t)$$

Unfortunately, we do not have a similar analytic solution for the three-body problem as we do for the two-body problem, meaning it can only be solved numerically. This can be done by integrating the equations of motion of the given problem, which are given by:

$$\ddot{r}_1=-Gm_2\frac{r_1-r_2}{|r_1-r_2|^3}-Gm_3\frac{r_1-r_3}{|r_1-r_3|^3}$$

$$\ddot{r}_2=-Gm_3\frac{r_2-r_3}{|r_2-r_3|^3}-Gm_1\frac{r_2-r_1}{|r_2-r_1|^3}$$

$$\ddot{r}_3=-Gm_1\frac{r_3-r_1}{|r_3-r_1|^3}-Gm_2\frac{r_3-r_2}{|r_3-r_2|^3}$$

Similar to the regular three body problem [4], the elliptic restricted three body problem is a type of three body problem which requires that two of the objects in the system have significantly larger masses than the third, (ie. M_1>>M_3 and M_2>>M_3), and that the third has a negligible gravitational influence on the primaries. Take (x_1, y_1) and (x_2, y_2) to be the position of the two massive bodies in cartesian coordinates. Then we get that the motion of the smallest mass is described by:

$$\ddot{x}=-m_1\frac{x-x_1}{((x-x_1)^2+(y-y_1)^2)^{3/2}}-m_2\frac{x-x_2}{((x-x_2)^2+(y-y_2)^2)^{3/2}}$$

$$\ddot{y}=-m_1\frac{y-y_1}{((x-x_1)^2+(y-y_1)^2)^{3/2}}-m_2\frac{y-y_2}{((x-x_2)^2+(y-y_2)^2)^{3/2}}$$

3. <u>Part one: the two-body problem</u>

Consider now a two-body orbital system where the mass of the first object far exceeds the mass of the second object. In our case the Sun and Earth. This leads to an orbital pattern where the effect of Earth's gravity on the Sun is negligible.

The two-body problem would usually be formulated in polar coordinates, however, we chose to use cartesian coordinates for this system due to the three-body problem, which is far easier to solve this way. We are given the equations of motion for this two-body problem to be:

$$\ddot{x} = -G M_s \frac{x}{(x^2+y^2)^{3/2}}$$

$$\ddot{y} = -G M_s \frac{y}{(x^2+y^2)^{3/2}},$$

Where M_s is the mass in solar masses, G is Newtons gravitational constant (taken as 58.9639 $AU^3/M_s Yr^2$) and x and y are the positional coordinates, measured in Astronomical Units.

Given the initial values of x and y as:

$$x_0 = 1.016714\,Au, y_0 = 0$$

We were tasked with finding the initial velocity of Earth's orbit such that we received an orbit with semimajor axis of 1.000000Au and eccentricity of 0.01671123.

We started with $V_y = 5$ and $V_y = 10$ and narrowed it down to get an approximate value of

$$V_y = 7.5545175$$

for which the perihelion is at 0.983286Au. This value will later be used in our three body. It is important to realize at what values our system fails. In particular at $V_y = 10$ or greater we get an unstable orbit due to Earths perihelion being outside the orbit of Jupiter and at $V_y < 4$ Earth's orbit would eventually fall into the Sun, giving us a range of possible initial velocities, as shown in Figure 1 below:

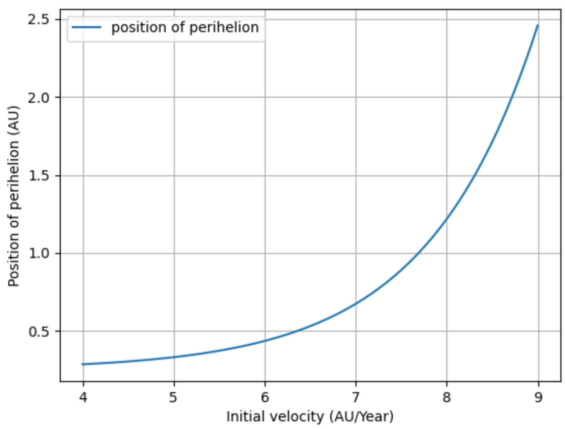

Figure 1: Trend of Perihelion position between $V_y = 4$ and $V_y = 9$

Using this initial velocity, we are able to plot Earth's 2-dimensional elliptical orbit around the Sun, shown in Figure 2.

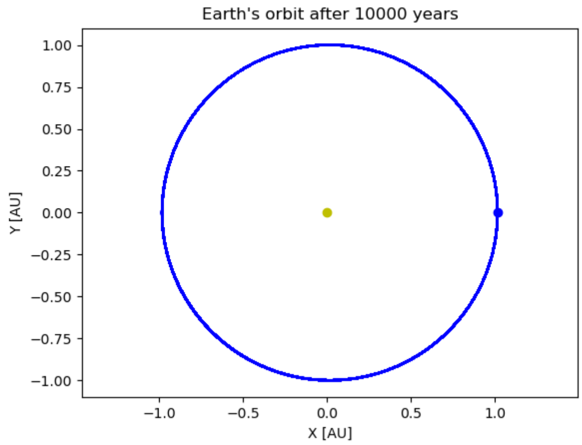

Figure 2: Earth's elliptical orbit with $V_y = 7.5545175$

Although figure 2 shows a stable graph of Earth's orbit, it became clear quickly that there was an inaccuracy with the given problem as the period of our orbit was roughly only 80% of a year. After doing further research we found the correct equation for G should have been:

$$G = 4\pi^2 AU^3 / M_s Yr^2 (\approx 39.4784) \text{ [5,7]}$$

This will be the G we use for all calculations going forward.

Thankfully this was an easy fix in the code and using the same procedure as before we found V_y for this G, narrowing it down to

$$V_y = 6.1795.$$

This gives us a stable orbit and an orbit period of almost one year, however we do still have a very small precession which becomes far more evident in our animation when speeds of integer multiples of 100 are taken. Similarly to before, we get a range of initial velocities for which we are able to get a stable orbit, where:

$$3\, Au/yr \leq V_y \leq 8\, Au/yr$$

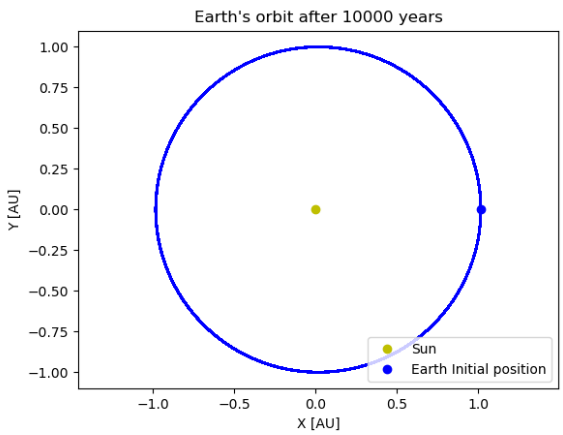

Figure 3: Earth's orbit around the sun with corrected G

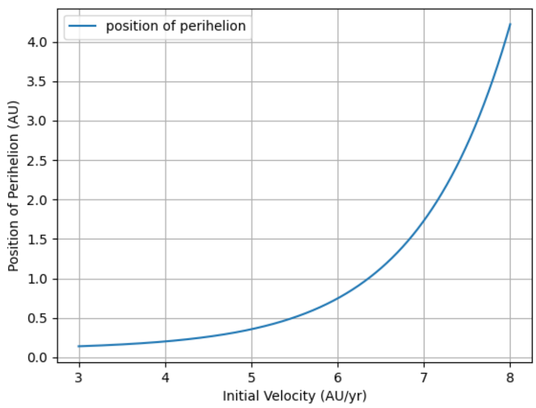

Figure 4: Trend of Perihelion between $V_y = 3$ and $V_y = 8$ with corrected G

The values of x and y at each point in time were found using SciPy's ODE integrator 'ODEINT', and was plotted using Matplotlib's module pyplot. It works by taking an array and coordinating the elements of the array which correspond with certain spots on the graph and plotting it. We could have built our own Eighth order Runge-Kutta Integrator to get a slightly more stable orbit throughout the simulation but found that the fourth order integrator in ODEINT did not produce any excessive rounding errors.

Although not noticeable for our time period, we did get slight variances in the stable orbit of Earth, resulting in an aphelion ranging between:

$$Aphelion_{max}=1.016945\ Au,\ Aphelion_{min}=1.016235\ Au,$$

and a perihelion ranging between:

$$Perihelion_{max}=0.983567\ Au,\ Perihelion_{min}=0.982344\ Au,$$

Over a larger time period however, this error would become far more noticeable and the step size would need to be decreased to combat this issue. This would drastically increase the computation time and thus we found that for a period of 10000 years an error of ± 0.0005 Au was acceptable.

Using the same equations as motion as before we also chose to plot Jupiter going around the Sun to allow us to see any possible changes Earth's gravitational field may have on Jupiter's orbit in the next section.

We are given the initial values of x and y as:

$$x_{j0}=0,\ y_{j0}=4.950429\ \text{Au}$$

It is stated in the brief that this is the aphelion with semimajor axis 5.204267, however this is incorrect as the aphelion cannot be less that semimajor axis by definition, thus we will take this y_{j0} as the perihelion of Jupiter's orbit.

Using the same method as before, we narrowed down the initial value of the Vx. For the incorrect value of G (58.9639), we get that:

$$V_x=-3.53463745\ Au/yr$$

and for the corrected value of G ($4\pi^2$), we get:

$$V_x=-2.8921\ Au/yr.$$

We get a range of possible initial speeds for which we get a stable orbit where:

$$1.5\ Au/yr \le V_x \le 3.5\ Au/yr$$

As shown in figure 6, but for any initial velocity less than 2 Au/yr there will be a significant effect on Jupiter's orbit.

Using the corrected initial velocity, we are able to plot Jupiter's 2-Dimensional orbit around the Sun shown in figure 5:

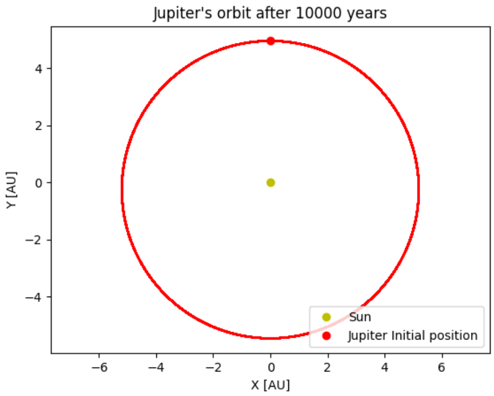

Figure 5: Jupiter's stable orbit around the sun

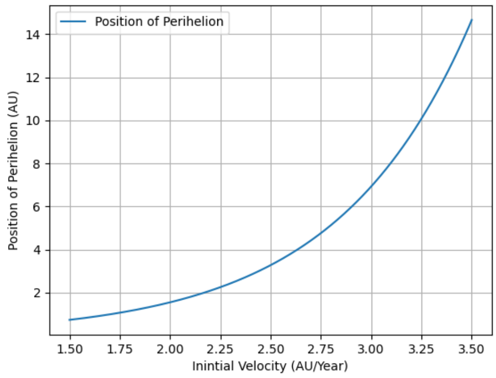

Figure 6: Trend of perihelion position between $V_x = 1.5$ and $V_x = 3.5$

Similar to with Earth's orbit, we have slight variances in each individual orbit of Jupiter. This resulted in an aphelion ranging between:

$$Aphelion_{max} = 5.458824\ Au\ ,\ Aphelion_{min} = 5.457581\ Au\ ,$$

and a perihelion ranging between:

$$Perihelion_{max} = 4.950429\ Au\ ,\ Perihelion_{min} = 4.949662\ Au\ ,$$

For our Perihelion our initial value is the outermost orbit, meaning that it never crosses back through. However, this leads to a more stable aphelion and thus we chose to stick with our Vx.

4. Part Two: The three-body system

For the three-body problem, after seeing the isolated Earth-Sun and Jupiter-Sun systems, we will now use a similar method to calculate the three-body problem and the effect that Earth and Jupiter have on each other's orbits.

We are given Earth's Equations of motion as:

$$\ddot{x}_E = -G M_s \frac{x_E}{\left(x_E^2 + y_E^2\right)^{3/2}} - G M_J \frac{x_E - x_J}{\left(\left(x_E - x_J\right)^2 + \left(y_E - y_J\right)^2\right)^{3/2}}$$

$$\ddot{y}_E = -G M_s \frac{y_E}{\left(x_E^2 + y_E^2\right)^{3/2}} - G M_J \frac{y_E - y_J}{\left(\left(x_E - x_J\right)^2 + \left(y_E - y_J\right)^2\right)^{3/2}}$$

And for Jupiter we are given:

$$\ddot{x}_J = -G M_s \frac{x_J}{\left(x_E^2 + y_E^2\right)^{3/2}} - G M_E \frac{x_E - x_J}{\left(\left(x_E - x_J\right)^2 + \left(y_E - y_J\right)^2\right)^{3/2}}$$

$$\ddot{y}_J = -G M_s \frac{y_J}{\left(x_E^2 + y_E^2\right)^{3/2}} - G M_E \frac{y_E - y_J}{\left(\left(x_E - x_J\right)^2 + \left(y_E - y_J\right)^2\right)^{3/2}}$$

In this case taking G to be the corrected gravitational constant ($4\pi^2$) with the mass of Earth (M_E) and the mass of Jupiter (M_J) given by:

$$M_E = \frac{1}{332950} M_S, M_J = \frac{1}{1048} M_S$$

Where Ms is the mass of the sun.

Once again using SciPi's ODEINT package, we used a multitude of arrays to integrate these equations of motion and the position at each time stamp was again plotted using Matplotlib's module pyplot. This gave us a graph of the three-body system and Jupiter and Earth's orbits relative to each other show below in Figure 7:

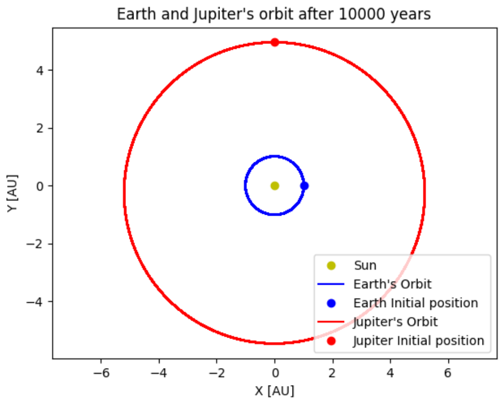

Figure 7: The Earth-Jupiter-Sun three body orbital system

Although usually associated with asteroids or comets and their orbits around two larger objects in the solar system, the system of Earth, Jupiter and the Sun shares some similarities with elliptic restricted three body problems [6]. In this case Earth has negligible affect on the shape of Jupiter's orbit around the Sun but Earth is significantly affected by Jupiter's gravitational field. We can make this effect far more apparent in the simulation by increasing Jupiter's mass.

Starting with a 10-fold increase in Jupiter's mass, you can start to see the change in Earth's orbit over a long period of time. Over our 10000-year simulation we get a far greater range in elliptical of orbits, accumulating into a near circular average orbit. The aphelion and perihelion flip back and forth continually meaning that we get a Right-side distance ranging between

$$R-distance_{max} = 1.023323\, Au, R-distance_{min} = 0.974037\, Au,$$

and a Left-side distance ranging between:

$$L-distance_{max} = 1.024569\, Au, L-distance_{min} = 0.9763117\, Au,$$

(Where before we took the Aphelion to be on the righthand side and the perihelion on the lefthand side). This leaves us with a range in eccentricity such that:

$$Eccentricity_{max} = 0.0235, Eccentricity_{min} = 6.12 \times 10^{-4},$$

To emphasize the point, we also decided to increase Jupiter's mass by 100 times its initial value, giving us values for the Right-side distance of

$$R-distance_{max}=1.10686198\,Au,\,R-distance_{min}=0.89180827\,Au,$$

and values for the Left-side distance of:

$$L-distance_{max}=1.1066136\,Au,\,L-distance_{min}=0.89173712\,Au,$$

And an eccentricity ranging between:

$$Eccentricity_{max}=0.1\,,Eccentricity_{min}=1.122\times10^{-4},$$

Neither of these situations give us a stable aphelion or perihelion to work with. Due to Jupiter's gravitational pull, Earth's orbit is continually shifting back and forth meaning that the Aphelion and Perihelion will be constantly changing. For the $10M_J$ the semi major axis (SMA) fluctuates between:

$$SMA_{max}=1.022697\,Au,\,SMA_{min}=0.973441\,Au,$$

Showing a clear change in the semi major axis. We get a similar result for the $100M_J$ where the SMA ranges between:

$$SMA_{max}=1.106\,Au,\,SMA_{min}=0.8107\,Au,$$

Which is a more pronounced change.

The instability in the orbit is much more evident when shown on a graph, as pictured below in Figure 8 and Figure 9.

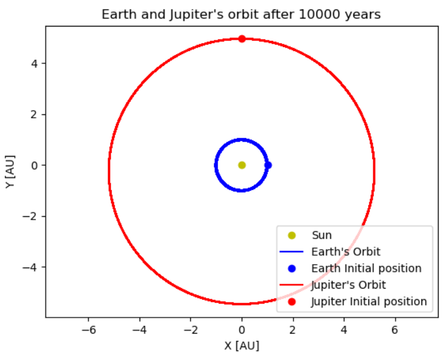

Figure 8: Earth-Jupiter-Sun three body system with $10M_J$

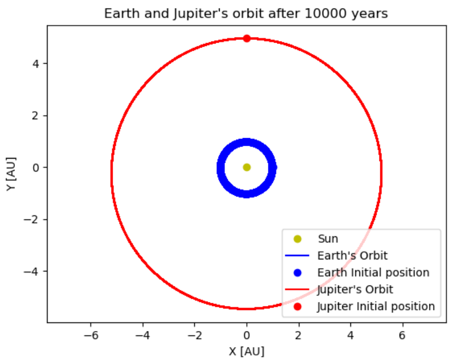

Figure 9: Earth-Jupiter-Sun three body system with $100M_J$

We also decided to run our original code for 100,000 years to see if there were any differences in the orbits of Jupiter or Earth. As predicted when talking about our error before, Earth's orbit is incredibly unstable with a large variance, such that there is no longer a stable aphelion or perihelion.

Jupiter's orbit remains mainly unchanged and in the EJS graph, the affect Jupiter has on the Earth is neglible compared to the compounding error and reduced step size as a result of our increased timescale.

This change is mainly due to our reduced step size of 0.1 when running the code for 100,000 years. You get the original results when running for 100,000 years and a step size of 0.1, however this code will take a very long time to load.

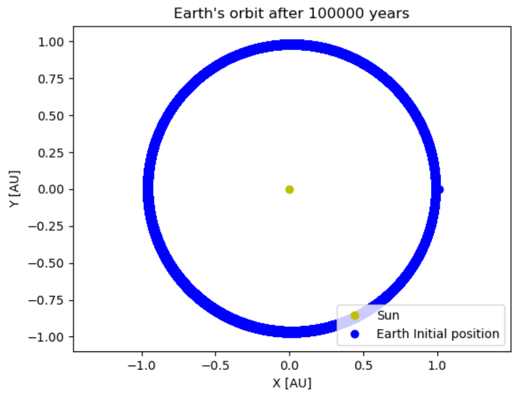

Figure 10: Elliptical orbit in our Earth-Sun system after 100000 years

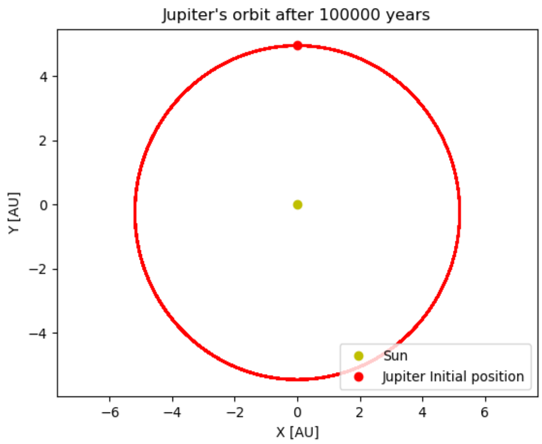

Figure 11: Elliptical orbit in our Jupiter-Sun system after 100000 years

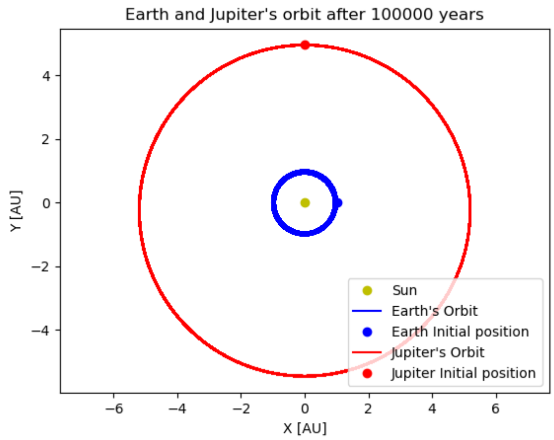

Figure 12: Elliptical orbit in our Earth-Jupiter-Sun system after 100000 years

The three-body system was animated using Matplotlib's animation packages, FuncAnimation and Animation. It works by repeatedly taking a graph of all the points and each frame plotting all the points in an arc that the body is moving in and then pasting the new graph on top of the previous one. By increasing the speed of the animation, you lower the number of points needing to be measured in each frame, thus giving the appearance of a moving point. There were some issues with getting the animation function to work, most of which involved Spyder not being able to handle the animation in its console window. We also encountered an overflow error when trying to animate the system over 100000 years. To fix this we introduced the package mpl.rcParams['agg.path.chunksize'] to exceed the cell block limit.

For every issue we also found some cool animations. When s = 81, the earth appears to be moving backwards. At s = 100, the Earth is practically stationary, slowly moving backwards. At s=1000 both Jupiter and Earth are moving backwards at nice speeds. We recommend playing around with s and see what happens!

5. <u>Further research</u>

As stated before, the Earth-Jupiter-Sun three body system bares many similarities to an elliptic restricted three body system [6]. We were interested in finding out how an increase in Jupiter's mass, such that it was similar to that of the Sun, would affect how the system behaved. This would lead to a system where Earth was essentially a comet orbiting between two near identical massive bodies.

We looked specifically at the case where the mass of Jupiter is:

$$M_J = \frac{9}{10} M_S$$

Creating a system where Earth is being pulled by near identical forces from either side. This creates a system where earth will fluctuate between a Jupiter orbit to a solar orbit as pictured below.

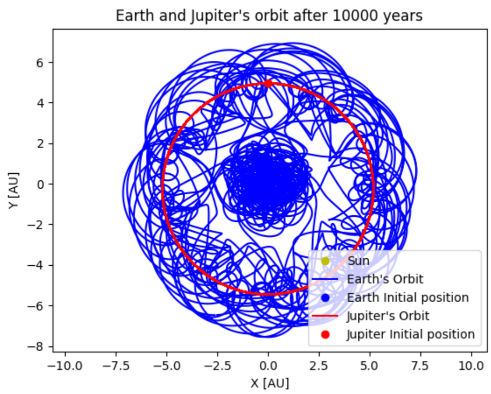

Figure 13: The Earth-Jupiter-Sun three body system where $M_J = 9/10\ M_S$

In our case with M_J being so large, the Sun and Jupiter should rotate together around their centre of mass, but unfortunately, our equations of motion only allow us to take the orbits of Jupiter and Earth relative to a stationary Sun.

We were also interested in seeing the effect that an increase in the Sun's mass would have on the orbits of Earth and Jupiter given the same initial conditions. Starting at 5 times the Sun's mass, we get far more elliptical orbits for Earth and Jupiter as shown below. The time period for these graphs have been shortened to show a more pronounced affect that the Sun has.

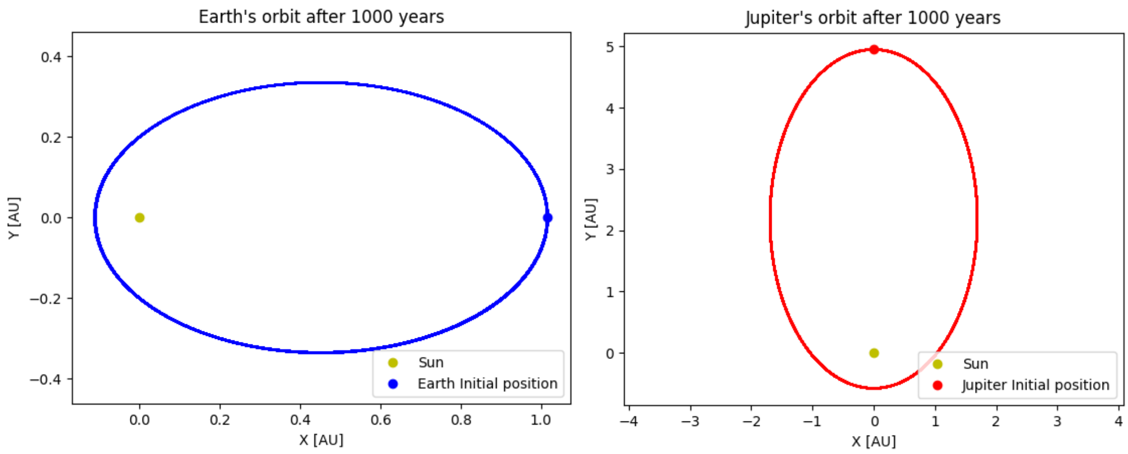

Figure 14 and 15: Earth and Jupiters orbits with $5\times M_S$ Respectively.

In the case above however we do not get anything spectacular for the three-body system, which is merely a near exact combination of the two singular orbits. For 10 times the Sun's mass on the other hand, we get a more interesting orbit pattern in the three-body-system:

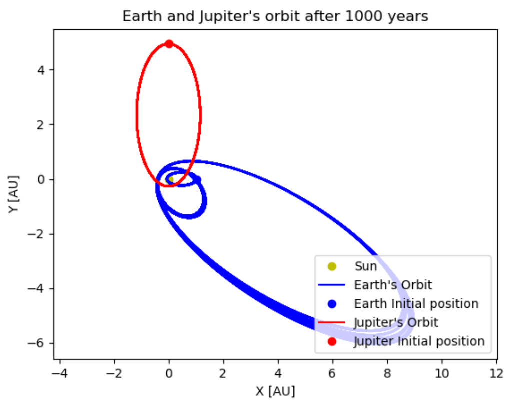

Figure 16: The Earth-Jupiter-Sun three body system with $10\times M_S$.

The increasing orbit size of Earth over time is because of its interaction with Jupiter. Due to the Sun's larger mass, it now has a greater gravitational pull, pulling Jupiter's orbit close enough such that it can now interact with Earths orbit. This in turn creates a system where Earth eventually gets close enough to Jupiter such that it is knocked out of its stable orbit.

6. Notes

Throughout our project, we came across a number of issues within the brief, most of which have already been noted throughout the report but will be summarized here. Our initial problem arose with the fact that the given G value was incorrect. This was easily fixed when we noticed that with the incorrect G, we could not get an Earth orbit which was close to 1 year. The correct value was taken to be:

$$G=4\pi^2$$

And was used throughout the rest of the report as the correct value for G. There were two smaller issues we found in the brief that initially went unnoticed but on closer inspection were found to be errors. Firstly, at the start of the three-body problem section, there is a sentence describing the initial conditions we should use for Jupiter, stating "You can start with Jupiter at its aphelion, y = 4.950429AU. The semimajor axis is 5.204267AU, and the eccentricity is 0.048775.". However, it is impossible to have an aphelion that is smaller than the semimajor axis, and rather should have been labelled the perihelion. Secondly, in equation (6) located in the three-body problem section, Jupiter's equation of motion for the y axis is given as:

$$\ddot{y}_J=-GM_s\frac{y_J}{\left(x_E^2+y_E^2\right)^{3/2}}-GM_E\frac{x_E-x_J}{\left(\left(x_E-x_J\right)^2+\left(y_E-y_J\right)^2\right)^{3/2}}$$

However, this is incorrect, and should have rather been given as:

$$\ddot{y}_J=-GM_s\frac{y_J}{\left(x_E^2+y_E^2\right)^{3/2}}-GM_E\frac{y_E-y_J}{\left(\left(x_E-x_J\right)^2+\left(y_E-y_J\right)^2\right)^{3/2}}$$

During the course of this project, we also encountered difficulty due to our programming platform Spyder. It became clear quite quickly that Spyder is not really adept at for larger more complicated problems, and it quite often struggled to run more complicated versions of our orbital system or failed when trying to calculate extensively long time periods. If we were to do this again or needed to modify the code for a more complicated system, we would look for a more suitable programming platform.

We also noticed while searching for our results that our code could in future quite easily be expanded to accommodate a larger orbital system. This for example would be useful if we wished to calculate how each element in the solar system interacted together. We did not attempt this however as we did not think Spyder had the computing power to do it, and it wouldn't have been able to be completed in the allocated timeframe.

7. Conclusion

Orbital three-body systems are an interesting subject which can tell us a lot about our planet and how it came to be. By its nature it is only solvable numerically, which in turn means that it lends itself well to a computational approach. Throughout this project we created a computational method for solving a three-body orbital problem rotating about a fixed mass with the help of a Runge-Kutta integrator. While our focus was on the Earth-Jupiter-Sun three-body system it should be clear that given the required data set, this method could be applied to any number of different three-body systems. Particularly for studying planetary orbits around the sun.

8. Acknowledgements

We would like to thank Peter Coles and Hannah O'Brennan for helping us figure out any issues with our code or with our understanding of the brief, and we would also like to thank the Maynooth University Theoretical Physics department for providing us with all possible assistance to help us get through this morbid degree.

9. References

[1] Edmond A. Mathez 2000, "Milutin Milankovitch: Seeking the Cause of the Ice Ages" *American Museum of Natural History website*, accessed 20/4/2023 <https://www.amnh.org/learn-teach/curriculum-collections/earth-inside-and-out/milutin-milankovitch-seeking-the-cause-of-the-ice-ages>

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